Kernel-Based Function Approximation for Average Reward Reinforcement Learning: An Optimist No-Regret Algorithm

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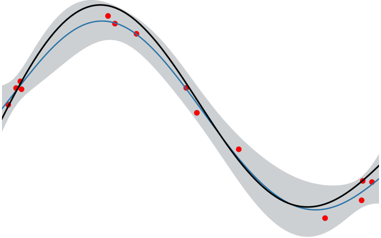
Overview

Kernel Based Regression

Provided a dataset of t observation:

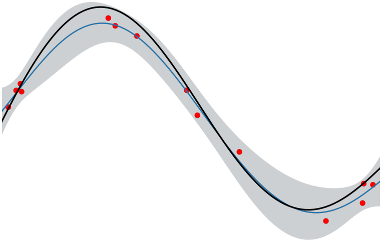
$$\left\{(z_j, Y(z_j))\right\}_{j=1}^{t}, Y(z_j) = f(z_j) + \varepsilon_j$$



Kernel-Based Regression

Predictor:

$$\hat{f}(z) = \boldsymbol{\kappa}_t^\top(z)(\mathbf{K}_t + \rho I)^{-1}\mathbf{y}_t$$

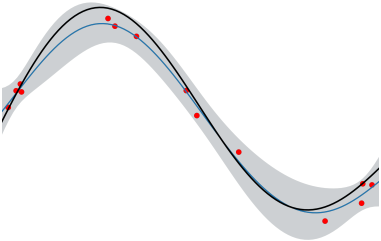


- $\boldsymbol{\kappa}_t(z) = [\kappa(z_1, z), \kappa(z_2, z), \cdots, \kappa(z_t, z)]$
- $\mathbf{K}_t = [\kappa(z_i, z_j)]_{i,j=1}^t$
- $\mathbf{y}_t = [Y(z_1), Y(z_2), \cdots, Y(z_t)]$

Kernel-Based Regression

Uncertainty estimator:

$$(\sigma_t(z))^2 = \kappa(z,z) - \boldsymbol{\kappa}_t^\top(z)(\mathbf{K}_t + \rho I)^{-1}\boldsymbol{\kappa}_t(z)$$



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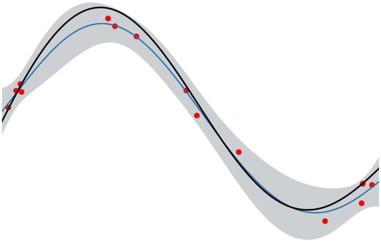

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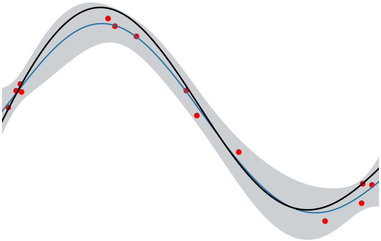

Closed from expressions for prediction and uncertainty quantification!

Confidence interval $|f(z) - \hat{f}_t(z)| \leq \beta(\delta)\sigma_t(z)$, w.p. $1 - \delta$

Bayesian and Frequentist Interpretations

Bayesian: Posterior mean (maximum likelihood estimation) assuming a prior centered Gaussian process distribution $\mathcal{GP}(\mathbf{0}, \kappa)$ and Gaussian noise

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Frequntist: Regularized Least Squares Error Estimation

$$\hat{f} = \arg\min_{g \in \mathcal{H}_\kappa} \sum_{j=1}^{t} \left(Y(z_j) - g(z_j)\right)^2 + \lambda \|g\|^2_{\mathcal{H}_\kappa}$$

Reproducing Kernel Hilbert Space

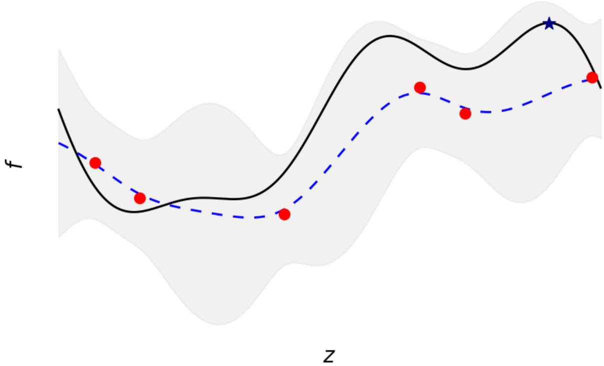
RKHS:

$$\mathcal{H}_\kappa = \{f(\cdot) = \sum_{m=1}^{\infty} w_m \phi_m(\cdot)\}$$

- Inner product $\langle f, g \rangle_{\mathcal{H}_\kappa} = \boldsymbol{w}_f^\top \boldsymbol{w}_g$
- $\|f\|_{\mathcal{H}_\kappa} = \|\mathbf{w}\|$
- $\phi_m = \sqrt{\lambda_m}\varphi_m$ form an orthonormal basis

Overview

Bandits (Bayesian Optimization)



Procedure: Sequentially select points $z_1, z_2, \cdots, z_T$

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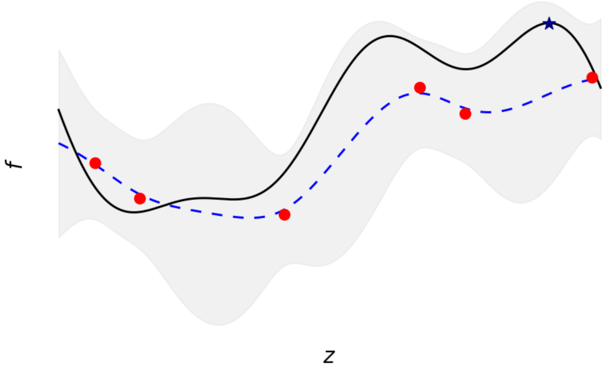

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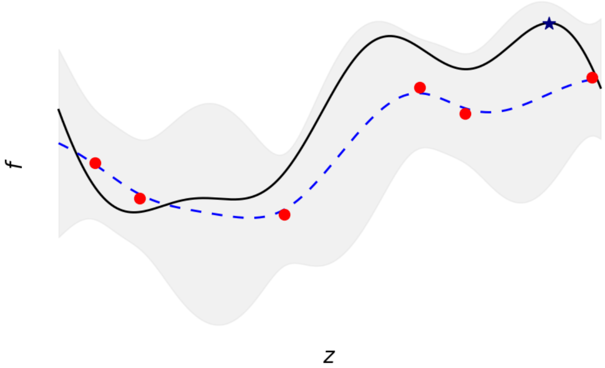

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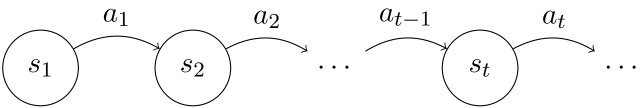
Optimistic Algorithm: $z_t = \arg\max_z \left(\hat{f}_{t-1}(z) + \beta(\delta)\sigma_{t-1}(z)\right)$

Overview

Markov Decision Process



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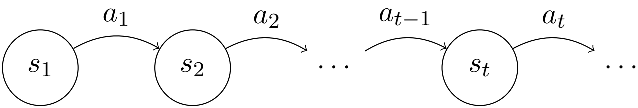

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$$J^\star = \max_\pi \liminf_{T\to\infty} \frac{1}{T}[\sum_{t=1}^{T} r(s_t, a_t)]$$

$$a_t = \pi(s_t), s_{t+1} \sim P(\cdot|s_t, a_t)$$

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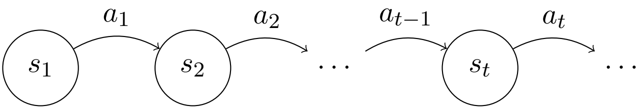

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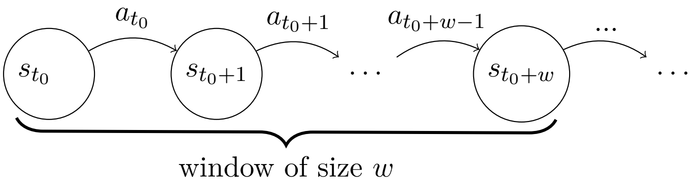
Weekly Communicating MDP

Value Function



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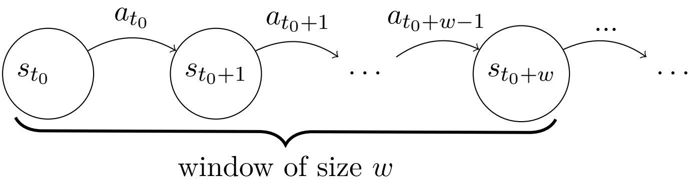

- The value of a policy π over a window w

$$v_w^\pi(s) = \mathbb{E}\left[\sum_{t=t_0}^{t_0+w-1} r(s_t, a_t)\right]$$

$$s = s_{t_0},\ a_t = \pi(s_t),\ s_{t+1} \sim P(\cdot|s_t, a_t)$$

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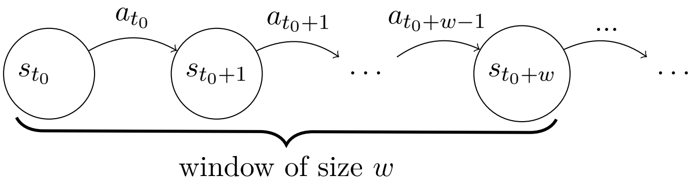

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$$q_w^\pi(s, a) = \mathbb{E}\left[\sum_{t=t_0}^{t_0+w-1} r(s_t, a_t)\right]$$

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Overview

Known Model

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$$[Pv](s,a) := \mathbb{E}_{s'\sim P(\cdot|s,a)}[v(s')] = \int_{s'\in\mathcal{S}} v(s')P(s'|s,a)ds'$$

- $v_{t_0+w} \leftarrow \mathbf{0}$
- Recursively $t = t_0 + w - 1, t_0 + w - 2, \cdots, t_0$:

$$q_t(s,a) = r(s,a) + [Pv_{t+1}](s,a)$$
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- $\pi_t^{\star,w}(s) = \arg\max_a q_t(s,a)$

Unknown Model : RL

r and P are unknown in RL

$$f_t = [Pv_{t+1}]$$
$$\text{UCB} = \hat{f}_t + \beta(\delta)\sigma_t$$

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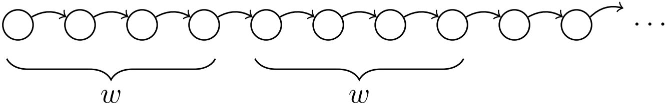
KUCB-RL Algorithm



- Fix a window size w
- For each window $t \in [t_0, t_0 + w - 1]$, compute q_t and v_t, recursively, starting from $v_{t_0+w} = \mathbf{0}$
- Unroll the policy $a_t = \arg\max_a q_t(s_t, a)$ over this window

This is equivalent to solving a w-step look ahead MDP

Overview

Confidence Intervals

- How to create confidence intervals for $f = [Pv]$

$$\left| \hat{f}_t(s,a) - [Pv](s,a) \right| \leq \beta(\delta)\sigma_t(s,a).$$

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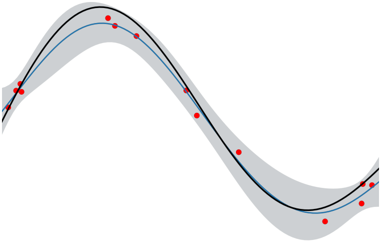
$$\left|\hat{f}_t(s,a) - [Pv](s,a)\right| \leq \beta(\delta)\sigma_t(s,a).$$

- For a fixed $f \in \mathcal{H}_\kappa$ with non-adaptive inputs $z_1, \ldots, z_t$,

$$\beta(\delta) \approx \|f\|_{\mathcal{H}_\kappa} + \frac{w}{\sqrt{\rho}}\sqrt{d\log(\tfrac{t}{\delta})}$$



Confidence Intervals

Challenge 1: Inputs $(s_1, a_1), \ldots, (s_t, a_t)$ are adaptive!

Solution: Self-normalized concentration inequalities for vector-valued martingales extended to kernel setting (Abbasi-Yadkori 2013; Whitehouse et al. 2023):

$$\beta(\delta) \approx \|f\|_{\mathcal{H}_\kappa} + \frac{w}{\sqrt{\rho}}\sqrt{\gamma(t;\rho) + \log(\tfrac{1}{\delta})}$$

- Maximum information gain: $\gamma(t;\rho) = \sup_{\{z_i\}_{i=1}^t \subset \mathcal{Z}} \frac{1}{2} \log\det\left(I + \frac{K_t}{\rho}\right)$

Confidence Intervals

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Solution: Optimistic closure assumption

- For all $v \in \mathcal{V}$ and some $\kappa' : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$, $\|v\|_{\mathcal{H}_{\kappa'}} \leq C_v = \mathcal{O}(w)$

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- Proof idea:
  - Write v in terms of the feature space of κ'
  - For M largest eigenvalues use standard confidence intervals (Whitehouse et al. 2023)
  - The rest is bounded using the eigendecay

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Assumption: $P(s|\cdot,\cdot)$ is in the RKHS $\mathcal{H}_k$ of a known kernel k

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Theorem: Regret bound w.p. $1 - \delta$

$$\mathcal{R}(T) = \mathcal{O}\left(\underbrace{\frac{T}{w}}_{w\text{-step look ahead}} + \underbrace{\beta(\delta) \sum_{t=1}^{T} \sigma_{w\lfloor \frac{t-1}{w} \rfloor}(s_t, a_t)}_{\text{Uncertainties in values}} \right)$$

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$$\mathcal{R}(T) = \mathcal{O}\left(\frac{T}{w} + \left(w + \frac{w}{\sqrt{\rho}}\sqrt{\gamma(T;\rho) + \log\left(\frac{T}{\delta}\right)} \right) \sqrt{\rho T \gamma(T;\rho) + \rho^2 w^2 \gamma(T;\rho)\gamma(T/w;\rho)} \right)$$

Result

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- For comparison, the regret bound for UCB algorithm in BO: $\mathcal{R}(T) = \tilde{\mathcal{O}}(T^{\frac{\nu+2d}{2\nu+2d}})$ (Whitehouse et al. 2023)

Discussion and Technical Challenges

- The policy is updated every w step: delay in using samples

$$\sum_{t=1}^{T} \sigma_{w\lfloor \frac{t-1}{w} \rfloor}(s_t, a_t)) \leq \sqrt{\rho T \gamma(T; \rho) + \rho^2 w^2 \gamma(T; \rho) \gamma(T/w; \rho)}$$

Elliptical potential lemma (Srinivas et al. 2010).

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- Confidence interval is not applied to prefixed functions (as in BO)

Summary

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Open problem: Is our bound improvable? What is the minimum rate of regret growth with T?

References I

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